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The impact of a thermoelastic rod against a rigid heated barrier

YU.A. ROSSIKHIN and M.V. SHITIKOVA
Department of Theoretical Mechanics, Voronezh State University of Architecture and Civil Engineering, ul. Kirova 3-75, Voronezh 394018, Russia. E-mail: shitikova@vmail.ru

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Abstract. The impact of a thermoelastic rod with a heat-insulated lateral surface against a rigid heated barrier is considered. The heat exchange between the rod and the wall occurs at one of its ends contacting with the wall, while the other end is heat-insulated and free from external forces. The behaviour of the rod during the impact process is described by the Green-Naghdy theory which allows one to take finite speed of heat propagation into account, neglecting therewith thermal relaxation. The Laplace integral transform with the subsequent expansion of the found images in terms of the natural functions of the problem is used as a method of solution, which is found in explicit exact closed form. The analytical time-dependence of displacements, stresses, and temperature at each rod particle is obtained. The emphasis is on the analysis of the contact stress, the temperature of the colliding bodies during their contact interaction, and on the detection of the duration of contact of the rod with the rigid wall. It is shown that the contact time essentially depends on the relationship between the mechanical and thermal values.

Key words: finite-speed heat propagation, Green-Naghdy theory, impact, thermoelastic rod

1. Introduction

Lord and Shulman [1] deduced the governing equations for extended thermoelasticity (ETE) based on the Maxwell–Cattaneo–Vernotte law

$$q = -\lambda\theta_{,x} - \tau\dot{q}, \tag{1}$$

where q is the heat-flux vector, $\theta = T - T_0$ is the relative temperature of a thermoelastic body, T is the temperature, T_0 is the temperature of the body in a natural state, $\lambda > 0$ is the thermal conductivity of the material, $\tau > 0$ is the relaxation time, an index after a comma denotes a derivative with respect to the x-coordinate, and an overdot is used for time differentiation.

The heat-transport equation based on the conduction law (1) is of hyperbolic type and predicts a finite speed for heat propagation, what has been experimentally verified for many liquid and solid materials (see review articles by Chandrasekharaiah [2, 3] and Joseph and Preziosi [4]).

The law of heat conduction (1) has two limiting cases, namely for $\tau = 0$ and $\tau \to \infty$. The former corresponds to the classical Fourier law

$$q = -\lambda\theta_{,x} \tag{2}$$

which results in an infinite speed for heat propagation what is physically unrealistic, particularly for initial-value problems and very short time intervals.

The second extreme limiting case ($\tau \to \infty$) governs the conduction law

$$\dot{q} = -\kappa\theta_{,x}\,, \qquad \kappa = \lim_{\tau\to\infty}\frac{\lambda}{\tau} \tag{3}$$

which admits undamped thermal waves propagating with a finite speed.

The governing equations of thermoelasticity based on the heat-conduction law (3) were derived and justified by Green and Naghdy in 1993 [5]. This new model of the thermoelasticity theory was defined as *thermoelasticity without energy dissipation* (hereafter referred to as TEWOED), since the internal rate of production of entropy has been put equal to zero.

If the first limiting case ($\tau = 0$) of ETE, corresponding to the conventional thermoelasticity (CTE), is applicable in situations where τ is very small compared to the time-scales involved, then the second limiting case ($\tau \to \infty$) of ETE, corresponding to TEWOED, is applicable in problems where τ is very large when compared to the time-scales. Rigid materials of this kind have already been identified [6].

Chandrasekharaiah in his state-of-the-art article [3] analyzing different models of the hyperbolic thermoelasticity has noted that, from a theoretical point of view, TEWOED is interesting in its own right, even when it is not viewed as a limiting case of ETE. In fact, TEWOED has been formulated independently of ETE; based on firm thermodynamical grounds, it can rightly be used as any other systematically developed thermomechanical theory. Green and Naghdy [5] say that this theory 'is perhaps a more natural candidate for its identification as *thermoelasticity*' than CTE.

TEWOED has immediately attracted considerable interest of researchers. A survey of the literature concerned with the theory can be found in [3]. Thus, in the context of TEWOED, free plane harmonic waves [7] and volume cylindrical/spherical waves [8–10] in an unbounded thermoelastic body, boundary-initiated waves in a half-space under different boundary loads [3, 11, 12], and Rayleigh waves in a half-space with stress-free and isothermal/adiabatic plane boundary [13] have been studied. It should be noted that in all these problems the explicit closed-form solutions have been constructed and analyzed.

In the present paper, the practically important problem about the collision of two solid bodies, one of which is considerably more rigid than the other, is solved by use of the Green-Naghdy theory [5]. The problem about the impact of a thermoelastic rod against a heated rigid wall fits naturally into the theory of TEWOED, since the process of contact interaction occurs so rapidly that the two bodies cease to be in contact with each other well before the thermal relaxation has had an opportunity to develop.

It should be mentioned that this boundary-value problem was considered previously in the context of ETE using the heat-conduction law (1) without reference to and with due account for the coupling of the strain and temperature fields in [14] and [15], respectively. The analysis of the solutions constructed in [14, 15] is amenable only to numerical treatment, whereas the application of the theory of TEWOED allows one to obtain the explicit exact solutions in closed form for the problem considered, as well as to analyze the influence of the thermal and mechanical parameters on the duration of contact of the rod with the wall.

2. Problem formulation

Let us consider a rod of length l with a heat-insulated lateral surface approaching a rigid barrier with the velocity v_0 (Figure 1). The wall temperature is T_1. Impact occurs at $t = 0$. The x-coordinate directed along the rod's axis is measured from the rod's section contacting with the wall. During impact, heat exchange between the rod and wall takes place on one end of the rod ($x = 0$); another end ($x = l$) is heat-insulated and free from external forces.

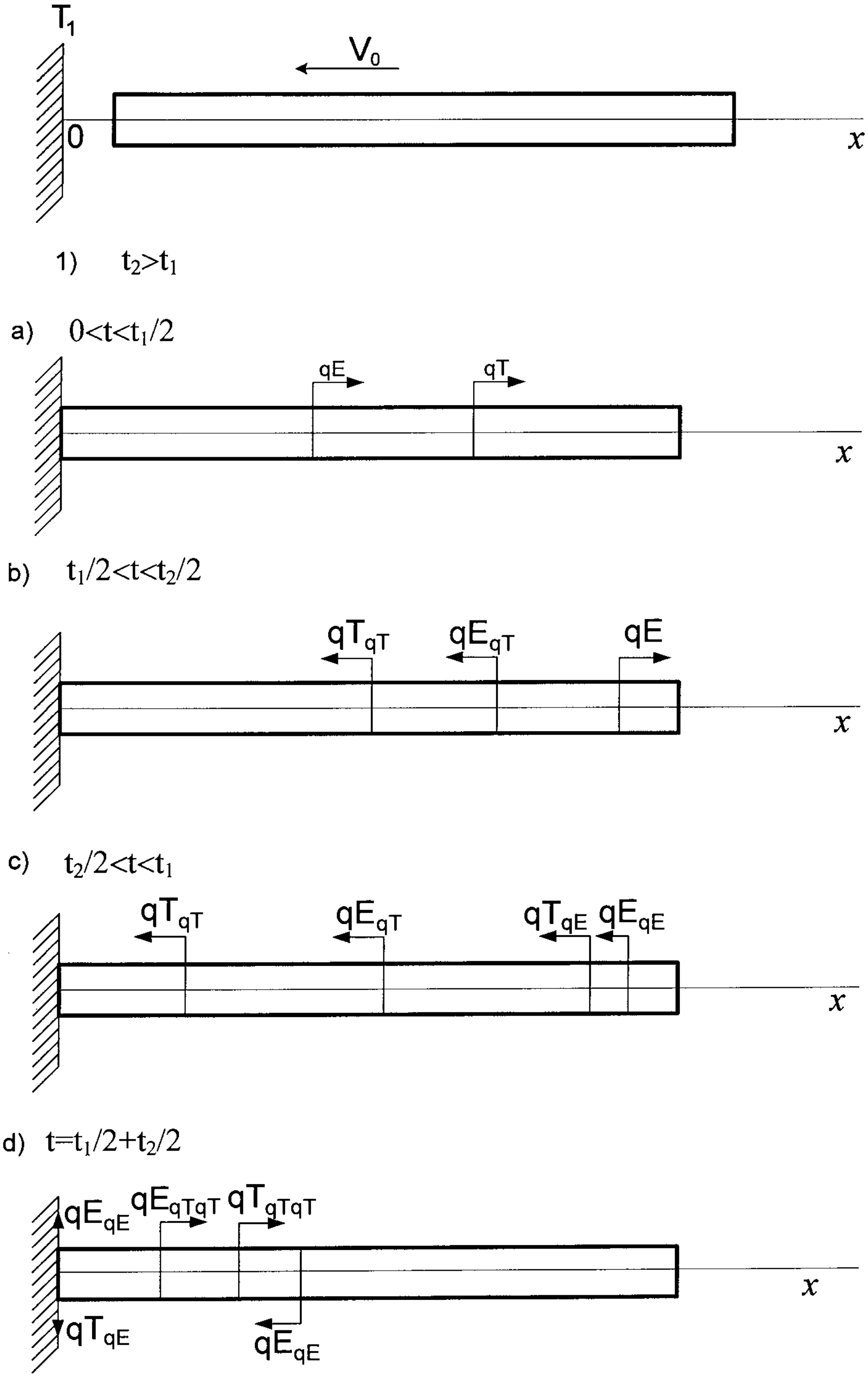

Figure 1. A scheme of a thermoelastic rod interaction with a rigid wall: 1) in the case $t_2 > t_1$ for $z = t_1/t_2 = 0{\cdot}6$ at the instants of time (a) $t = t_1/3$, (b) $t = 0{\cdot}75t_1$, (c) $t = 0{\cdot}9t_1$, and (d) $t = t_1/2 + t_2/2 = 1{\cdot}335t_1$; 2) in the case $t_1 > t_2$ for $z = 1{\cdot}4$ at the instants of time (a) $t = 0{\cdot}3t_2$, (b) $t = 0{\cdot}6t_2$, (c) $t = 0{\cdot}8t_2$, and (d) $t = t_2$.

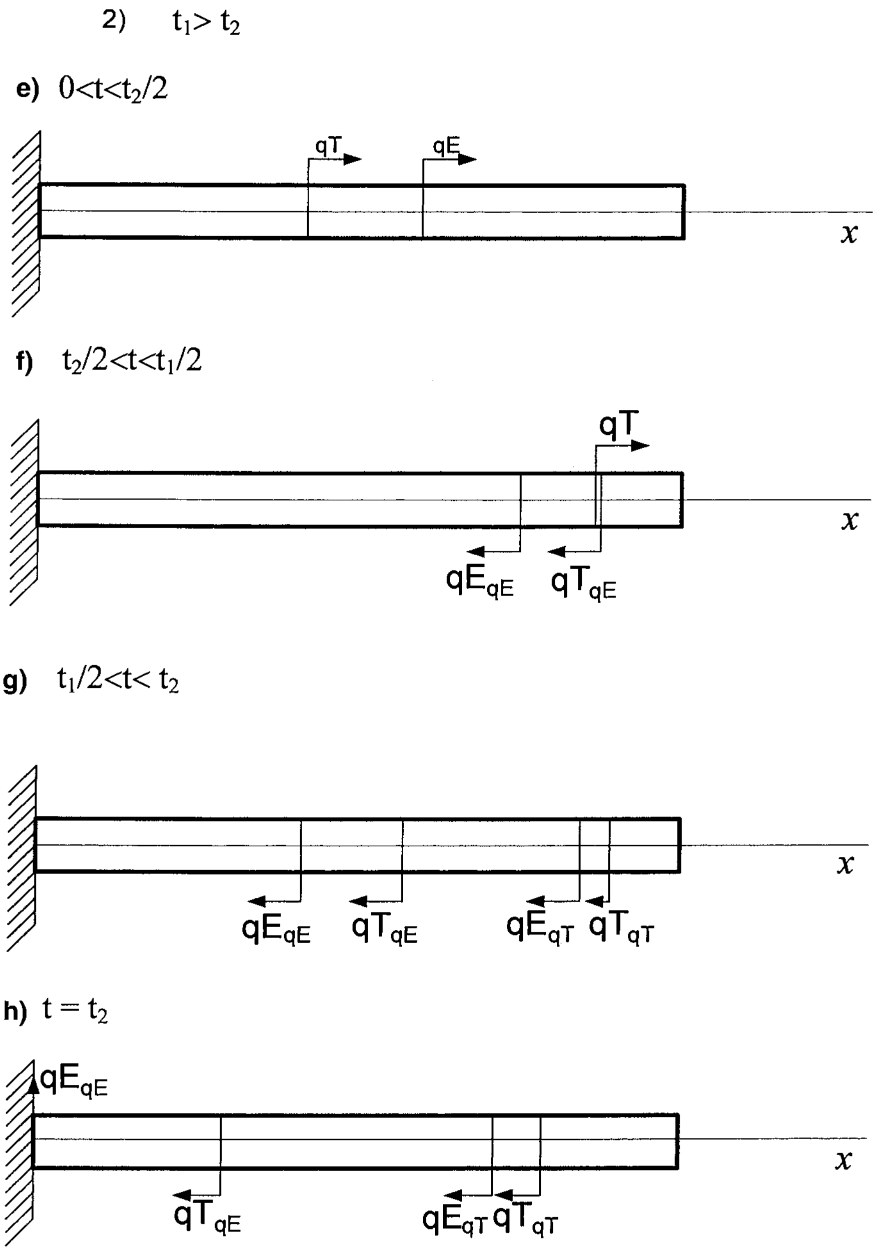

Figure 1. Continued.

Assuming that heat propagates with a finite speed and using the TEWOED theory, we come to the following problem: to find the solution of the set of equations

$$\ddot{\theta} - a^2\theta,_{xx} + r\gamma\ddot{u},_x = 0, \tag{4a}$$

$$\ddot{u} - c^2 u,_{xx} + c^2\alpha\theta,_x = 0 \tag{4b}$$

subjected to the initial and boundary conditions

$$u = 0, \quad \dot{u} = -v_0, \quad \theta = 0, \quad \dot{\theta} = 0 \quad (t = 0), \tag{5}$$

$$u = 0, \quad -q = h(\theta - \Theta_1) \quad (x = 0), \tag{6a}$$

$$\sigma = Eu_{,x} - \gamma\theta = 0, \quad q = 0 \quad (x = l), \tag{6b}$$

where $\theta = T - T_0$ is the relative temperature of the rod, T_0 is the temperature of the rod in a natural state, $a = \sqrt{\kappa/c_\varepsilon}$ is the thermal wave velocity, $r = T_0 c_\varepsilon^{-1}$, c_ε is the specific heat at constant strain, $\gamma = E\alpha$, E is the Young's modulus, α is the coefficient of linear thermal expansion, u is the displacement, $c = \sqrt{E/\rho}$ is the elastic wave velocity, ρ is the density, q is the quantity of heat flowing through a rod cross-section area unit per a time unit, σ is the stress, h is the coefficient of convective heat exchange from the wall to the rod, and $\Theta_1 = T_1 - T_0$.

At the moment of impact two shock waves originate at the contacting section of the rod, which then propagate along the rod with certain velocities c_1 and c_2 (Figure 1a). One of these waves will be called 'quasielastic' and denoted by qE, and the other will be termed 'quasithermal' wave and denoted by qT. Each of these waves has a mixed character, *i.e.* it possesses both elastic and thermal features; in so doing the elastic properties dominate over the qE-wave and the thermal features dominate over the qT-wave. In the case of ignoring the coupling of strain and temperature fields, these waves go over into pure elastic waves and pure thermal waves, respectively.

Assume that the qT-wave is faster than the qE-wave (see the first example in Figure 1). Then first the qT-wave reaches the free end of the rod, and two reflected waves appear at a time (Figure 1b), namely: the qT_{qT}-wave (the reflected quasithermal wave from the incident qT-wave) and the qE_{qT}-wave (the reflected quasielastic wave from the incident qT-wave). When the qE-wave comes up closer to the free end of the rod, two additional reflected waves appear (Figure 1c): the qT_{qE}-wave (the reflected quasithermal wave from the incident qE-wave) and the qE_{qE}-wave (the reflected quasielastic wave from the incident qE-wave).

First, the reflected qT_{qT}-wave comes to the contact region (the time of its arrival is $t_1 = 2lc_1^{-1}$), then at once two waves, the qT_{qE}- and qE_{qT}-waves, reach the contact zone (the time of their arrival is equal to $lc_1^{-1} + lc_2^{-1}$), and finally, the qE_{qE}-wave arrives at the contact region (the time of its arrival is $t_2 = 2lc_2^{-1}$). Each of these waves changes the contact stress $\sigma(0, t)$ abruptly, which may lead to the rebound of the rod from the rigid wall, if $\sigma(0, t)$ changes a sign. Thus, the contact conditions (6a) may break down at one of the enumerated instants of the time, and hence the problem formulation must include the condition $\sigma(0, t) \leq 0$ (the condition of loading of the section being in contact) which is valid only on the restricted time interval.

When the qE-wave is faster than the qT-wave (see the second example in Figure 1), then first the qE-wave reaches the free end of the rod generating two reflected waves: the qE_{qE}- and qT_{qE}-waves (Figure 1f). When the off-loading qE_{qE}-wave reaches the place of contact (Figure 1h), the contact stress changes abruptly to zero, resulting in the rod's rebound from the wall.

3. Method of solution

As a method of solution of the problem under consideration we shall use the Laplace-integral-transform method in combination with the expansion of the desired functions in terms of eigenfunctions. Applying the Laplace transformation to the set of Equations (4) and considering the initial and boundary conditions (5) and (6), we have

$$p^2\bar{\theta} - a^2\bar{\theta}_{,xx} + \varepsilon\alpha^{-1}p^2\bar{u}_{,x} = 0, \tag{7a}$$

$$p^2\bar{u} - c^2\bar{u},_{xx} + c^2\alpha\bar{\theta},_x = -v_0, \tag{7b}$$

$$\bar{u} = 0, \quad \kappa \frac{1}{p}\,\bar{\theta},_x = h\left(\bar{\theta} - \frac{\Theta_1}{p}\right) \quad (x = 0), \tag{8a}$$

$$E\bar{u},_x - \gamma\bar{\theta} = 0, \quad \frac{1}{p}\,\bar{\theta},_x = 0 \quad (x = l), \tag{8b}$$

where a bar over a function denotes the Laplace transform of the corresponding function, p is the complex parameter of the Laplace transformation, and $\varepsilon = r\alpha\gamma$ is the dimensionless parameter characterizing coupling of strain and temperature fields [16].

Let us seek the solution of the set of Equations (7) satisfying the boundary conditions (8) in the form of the superposition of four particular solutions of the following form:

$$\bar{u}_1 = \sum_{n=1}^{\infty} \bar{M}_n \cos l_n x, \quad \bar{\theta}_1 = \sum_{n=1}^{\infty} \bar{\chi}_n \sin l_n x, \tag{9a}$$

$$\bar{u}_2 = \sum_{n=1}^{\infty} \bar{G}_n \cos l_n x + \bar{g}\,\frac{x}{E}, \quad \bar{\theta}_2 = \sum_{n=1}^{\infty} \bar{\psi}_n \sin l_n x, \tag{9b}$$

$$\bar{u}_3 = \sum_{n=1}^{\infty} \bar{\Gamma}_n \cos l_n x + \bar{f}, \quad \bar{\theta}_3 = \sum_{n=1}^{\infty} \bar{\varphi}_n \sin l_n x, \tag{9c}$$

$$\bar{u}_4 = \sum_{n=1}^{\infty} \bar{N}_n \cos l_n x, \quad \bar{\theta}_4 = \sum_{n=1}^{\infty} \bar{c}_n \sin l_n x + \bar{s}, \tag{9d}$$

where $\bar{M}_n$, $\bar{G}_n$, $\bar{\Gamma}_n$, $\bar{N}_n$, $\bar{\chi}_n$, $\bar{\psi}_n$, $\bar{\varphi}_n$, $\bar{c}_n$, $\bar{g}$, $\bar{f}$, and $\bar{s}$ are yet unknown functions of the variable p to be determined from the set of Equations (7) and the boundary conditions (8), and $l_n = (2n-1)\pi(2l)^{-1}$.

The particular solution (9a) is the solution of the nonhomogeneous system of Equations (7), but other particular solutions (9b)–(9d) are the solutions of the corresponding homogeneous set of equations. Note that under such a choice of solution the second condition from (8b) is fulfilled automatically.

Substituting sequentially the expressions (9a)–(9d) in the set of Equations (7), and using the conditions of orthogonality for $\sin l_n x$ and $\cos l_n x$ on the segment from 0 to l, we find

$$\bar{M}_n = -\frac{2v_0(-1)^{n-1}}{l l_n}\,\frac{p^2 + a^2 l_n^2}{f(p)}, \quad \bar{\chi}_n = -\varepsilon\,\frac{2v_0(-1)^{n-1}}{l\alpha}\,\frac{p^2}{f(p)}, \tag{10a}$$

$$\begin{aligned}\bar{G}_n &= -\frac{2}{E l l_n^2}\frac{\left\{\left[l l_n(-1)^{n-1} - 1\right](p^2 + a^2 l_n^2) - \varepsilon c^2 l_n^2\right\} p^2 \bar{g}}{f(p)},\\ \bar{\psi}_n &= -\varepsilon\,\frac{2}{E\alpha l}\,\frac{[l(-1)^{n-1}p^2 + c^2 l_n]p^2\bar{g}}{f(p)},\end{aligned} \tag{10b}$$

$$\bar{\Gamma}_n = -\frac{2(-1)^{n-1}}{l l_n}\,\frac{p^2(p^2 + a^2 l_n^2)\bar{f}}{f(p)}, \quad \bar{\varphi}_n = -\varepsilon\,\frac{2(-1)^{n-1}}{l\alpha}\,\frac{p^4\bar{f}}{f(p)}, \tag{10c}$$

$$\bar{N}_n = \frac{2\alpha c^2}{l}\,\frac{p^2\bar{s}}{f(p)}, \quad \bar{c}_n = -\frac{2}{ll_n}\,\frac{p^2(p^2+c^2l_n^2)\bar{s}}{f(p)}, \tag{10d}$$

where $f(p) = (p^2 + c^2l_n^2)(p^2 + a^2l_n^2) + \varepsilon c^2l_n^2p^2$.

Going in the expressions (10) from images to pre-images, we obtain

$$\begin{aligned}
M_n(t) &= -\frac{2v_0(-1)^{n-1}}{ll_n(\Omega_n^2-\omega_n^2)}\left(\frac{\Omega_n^2-a^2l_n^2}{\Omega_n}\,\sin\Omega_n t - \frac{\omega_n^2-a^2l_n^2}{\omega_n}\,\sin\omega_n t\right), \\
\chi_n(t) &= -\varepsilon\,\frac{2v_0(-1)^{n-1}}{l\alpha(\Omega_n^2-\omega_n^2)}\,(\Omega_n\sin\Omega_n t - \omega_n\sin\omega_n t),
\end{aligned} \tag{11a}$$

$$\begin{aligned}
G_n(t) &= \frac{2[ll_n(-1)^{n-1}-1]}{Ell_n^2}\left\{-g(t) + \frac{1}{\Omega_n^2-\omega_n^2}\int_0^t g(t')\right. \\
&\quad\times\big[\Omega_n(\Omega_n^2-a^2l_n^2)\sin\Omega_n(t-t') \\
&\quad\left.-\,\omega_n(\omega_n^2-a^2l_n^2)\sin\omega_n(t-t')\big]\,\mathrm{d}t'\right\} + \varepsilon\,\frac{2c^2l_n^2}{Ell_n^2(\Omega_n^2-\omega_n^2)} \\
&\quad\times\int_0^t g(t')\big[\Omega_n\sin\Omega_n(t-t') - \omega_n\sin\omega_n(t-t')\big]\,\mathrm{d}t',
\end{aligned} \tag{11b}$$

$$\begin{aligned}
\psi_n(t) &= \varepsilon\,\frac{2(-1)^{n-1}}{E\alpha}\left\{-g(t) + \frac{1}{\Omega_n^2-\omega_n^2}\int_0^t g(t')\right. \\
&\quad\left.\times\big[\Omega_n^3\sin\Omega_n(t-t') - \omega_n^3\sin\omega_n(t-t')\big]\,dt'\right\} \\
&\quad-\varepsilon\,\frac{2c^2l_n^2}{E\alpha ll_n(\Omega_n^2-\omega_n^2)}\times\int_0^t g(t')\big[\Omega_n\sin\Omega_n(t-t') - \omega_n\sin\omega_n(t-t')\big]\,\mathrm{d}t',
\end{aligned}$$

$$\begin{aligned}
\Gamma_n(t) &= \frac{2(-1)^{n-1}}{ll_n}\left\{-f(t) + \frac{1}{\Omega_n^2-\omega_n^2}\int_0^t f(t')\right. \\
&\quad\left.\times\big[\Omega_n(\Omega_n^2-a^2l_n^2)\sin\Omega_n(t-t') - \omega_n(\omega_n^2-a^2l_n^2)\sin\omega_n(t-t')\big]\,\mathrm{d}t'\right\} \\
\varphi_n(t) &= \varepsilon\frac{2(-1)^{n-1}}{l\alpha}\left\{-f(t) + \frac{1}{\Omega_n^2-\omega_n^2}\int_0^t f(t')\right. \\
&\quad\left.\times\big[\Omega_n^3\sin\Omega_n(t-t') - \omega_n^3\sin\omega_n(t-t')\big]\,\mathrm{d}t'\right\},
\end{aligned} \tag{11c}$$

$$\begin{aligned}
N_n(t) &= \frac{2\alpha c^2}{l(\Omega_n^2-\omega_n^2)}\int_0^t s(t')\big[\Omega_n\sin\Omega_n(t-t') - \omega_n\sin\omega_n(t-t')\big]\,\mathrm{d}t', \\
c_n(t) &= \frac{2}{ll_n}\left\{-s(t) + \frac{1}{\Omega_n^2-\omega_n^2}\int_0^t s(t')\big[\Omega_n(\Omega_n^2-c^2l_n^2)\sin\Omega_n(t-t')\right. \\
&\quad\left.-\,\omega_n(\omega_n^2-c^2l_n^2)\sin\omega_n(t-t')\big]\,\mathrm{d}t'\right\},
\end{aligned} \tag{11d}$$

where $p^2 = -\omega_n^2$ and $p^2 = -\Omega_n^2$ are the roots of the characteristic equation $f(p) = 0$, $\omega_n = l_n 2lt_1^{-1}$, $\Omega_n = l_n 2lt_2^{-1}$, $t_1 = 2lc_1^{-1}$, $t_2 = 2lc_2^{-1}$, and the velocities of the $qT-$ and $qE-$ waves are defined from

$$c_{1,2}^2 = \frac{a^2 + c^2(1+\varepsilon)}{2} \pm \frac{1}{2}\sqrt{(a^2-c^2)^2 + c^2\varepsilon[2a^2 + c^2(2+\varepsilon)]}. \tag{12}$$

Writing the desired solution of the set of Equations (7) in the form of the superposition of the particular solutions (9)

$$\bar{u} = \sum_{i=1}^{4} \bar{u}_i, \qquad \bar{\theta} = \sum_{i=1}^{4} \bar{\theta}_i, \tag{13}$$

and substituting (13) in the boundary conditions (8), we have

$$\sum_{n=1}^{\infty}\left(\bar{M}_n + \bar{G}_n + \bar{\Gamma}_n + \bar{N}_n\right) + \bar{f} = 0, \tag{14a}$$

$$\frac{\kappa}{p}\sum_{n=1}^{\infty} l_n\left(\bar{\chi}_n + \bar{\psi}_n + \bar{\varphi}_n + \bar{c}_n\right) = h\left(\bar{s} - \frac{\Theta_1}{p}\right), \tag{14b}$$

$$\begin{aligned} & E\sum_{n=1}^{\infty} l_n(-1)^{n-1}\left(\bar{M}_n + \bar{G}_n + \bar{\Gamma}_n + \bar{N}_n\right) - \bar{g}, \\ & +\gamma\sum_{n=1}^{\infty}(-1)^{n-1}\left(\bar{\chi}_n + \bar{\psi}_n + \bar{\varphi}_n + \bar{c}_n\right) + \gamma\bar{s} = 0. \end{aligned} \tag{14c}$$

Going in Equations (14) from images to pre-images, we obtain the set of three equations for determining the three unknown functions: $f(t)$, $s(t)$, and $g(t)$

$$\begin{aligned} & -\frac{2v_0}{l}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{l_n(\Omega_n^2-\omega_n^2)}\left(\frac{\Omega_n^2 - a^2l_n^2}{\Omega_n}\ \sin\Omega_n t - \frac{\omega_n^2 - a^2 l_n^2}{\omega_n}\ \sin\omega_n t\right) \\ & + \sum_{n=1}^{\infty}\frac{2[ll_n(-1)^{n-1} - 1]}{Ell_n^2(\Omega_n^2-\omega_n^2)}\int_0^t g(t')\left[\Omega_n(\Omega_n^2 - a^2l_n^2)\sin\Omega_n(t-t')\right. \\ & \left. - \omega_n(\omega_n^2 - a^2l_n^2)\sin\omega_n(t-t')\right]\mathrm{d}t' + \varepsilon\,\frac{2c^2}{El}\sum_{n=1}^{\infty}\frac{1}{\Omega_n^2-\omega_n^2} \\ & \times\int_0^t g(t')\left[\Omega_n\sin\Omega_n(t-t') - \omega_n\sin\omega_n(t-t')\right]\mathrm{d}t' + \frac{2}{l}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{l_n(\Omega_n^2-\omega_n^2)}\int_0^t f(t') \\ & \times\left[\Omega_n(\Omega_n^2 - a^2l_n^2)\sin\Omega_n(t-t') - \omega_n(\omega_n^2-a^2l_n^2)\sin\omega_n(t-t')\right]\mathrm{d}t' \\ & +\frac{2\alpha c^2}{l}\sum_{n=1}^{\infty}\frac{1}{\Omega_n^2-\omega_n^2}\ \int_0^t s(t')\left[\Omega_n\sin\Omega_n(t-t') - \omega_n\sin\omega_n(t-t')\right]\mathrm{d}t' = 0, \end{aligned} \tag{15a}$$

$$
\begin{aligned}
&-\frac{hl\gamma}{2\kappa}\,[s(t)-\Theta_1]+\varepsilon v_0 E\sum_{n=1}^{\infty}\frac{l_n(-1)^{n-1}}{\Omega_n^2-\omega_n^2}\,[\cos\Omega_n t-\cos\omega_n t]\\
&+\varepsilon c^2\sum_{n=1}^{\infty}\frac{l_n^2}{\Omega_n^2-\omega_n^2}\int_0^t g(t')\left[\cos\Omega_n(t-t')-\cos\omega_n(t-t')\right]\mathrm{d}t'\\
&-\varepsilon\sum_{n=1}^{\infty}\frac{ll_n(-1)^{n-1}}{\Omega_n^2-\omega_n^2}\int_0^t g(t')\left[\Omega_n^2\cos\Omega_n(t-t')-\omega_n^2\cos\omega_n(t-t')\right]\mathrm{d}t'\\
&-\varepsilon E\sum_{n=1}^{\infty}\frac{l_n(-1)^{n-1}}{\Omega_n^2-\omega_n^2}\int_0^t f(t')\left[\Omega_n^2\cos\Omega_n(t-t')-\omega_n^2\cos\omega_n(t-t')\right]\mathrm{d}t'\\
&-\sum_{n=1}^{\infty}\frac{\gamma}{\Omega_n^2-\omega_n^2}\int_0^t s(t')\left[(\Omega_n^2-c^2l_n^2)\cos\Omega_n(t-t')-(\omega_n^2-c^2l_n^2)\cos\omega_n(t-t')\right]\mathrm{d}t',
\end{aligned}
\tag{15b}
$$

$$
\begin{aligned}
g(t)\ &=-\frac{2v_0E}{l}\sum_{n=1}^{\infty}\frac{1}{\Omega_n^2-\omega_n^2}\left(\frac{\Omega_n^2-a^2l_n^2}{\Omega_n}\ \sin\Omega_n t-\frac{\omega_n^2-a^2l_n^2}{\omega_n}\ \sin\omega_n t\right)\\
&+\sum_{n=1}^{\infty}\frac{2[(-1)^{n-1}-ll_n]}{ll_n(\Omega_n^2-\omega_n^2)}\int_0^t \dot g(t')\left[(\Omega_n^2-a^2l_n^2)\cos\Omega_n(t-t')\right.\\
&\left.-\,(\omega_n^2-a^2l_n^2)\cos\omega_n(t-t')\right]\mathrm{d}t'-\sum_{n=1}^{\infty}\frac{2E}{l(\Omega_n^2-\omega_n^2)}\int_0^t \dot f(t')\\
&\times\left[(\Omega_n^2(1+\varepsilon)-a^2l_n^2)\cos\Omega_n(t-t')-(\omega_n^2(1+\varepsilon)-a^2l_n^2)\cos\omega_n(t-t')\right]\mathrm{d}t'\\
&-\varepsilon\,\frac{2v_0E}{l}\sum_{n=1}^{\infty}\frac{1}{\Omega_n^2-\omega_n^2}\,(\Omega_n\sin\Omega_n t-\omega_n\sin\omega_n t)\\
&-\varepsilon\sum_{n=1}^{\infty}\frac{2}{\Omega_n^2-\omega_n^2}\int_0^t \dot g(t')\left[\Omega_n^2\cos\Omega_n(t-t')-\omega_n^2\cos\omega_n(t-t')\right]\mathrm{d}t'\\
&+\sum_{n=1}^{\infty}\frac{2\gamma(-1)^{n-1}}{ll_n(\Omega_n^2-\omega_n^2)}\int_0^t s(t')\left[\Omega_n^3\sin\Omega_n(t-t')-\omega_n^3\sin\omega_n(t-t')\right]\mathrm{d}t'.
\end{aligned}
\tag{15c}
$$

3.1. THE CASE $t_2 > t_1$

First we consider the more interesting case when the quasithermal wave is faster than the quasilongitudinal wave, *i.e.* $t_2 > t_1$ (see the first example in Figure 1). Then, with due account for the Fourier series presented in Appendix A, the set of Equations (15) gives

$$
\gamma s(t)=\frac{1}{\Delta}\left\{\Theta_1\gamma d_1+\varepsilon\ v_0E\ \frac{\kappa}{hc_1c_2}\right\},\quad (0<t<t_1), \tag{16a}
$$

$$\begin{aligned}\gamma s(t) &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma d_1 - \varepsilon\, v_0 E\,\frac{\kappa}{hc_1c_2}\\ &+\frac{2d_1\kappa}{hc_1(c_2^2-c_1^2)}\left[\varepsilon c^2 g(t-t_1)+(c^2-c_1^2)\gamma s(t-t_1)\right]\\ &+\varepsilon\frac{\kappa}{hc_2(c_2^2-c_1^2)}\left[(c^2-c_2^2)g(t-t_1)-c^2\gamma s(t-t_1)\right]\\ &-\varepsilon\,\frac{2E\kappa}{hc_1c_2(c_2^2-c_1^2)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right]\\ &-\varepsilon\,\frac{2Ed_1\kappa}{h(c_2^2-c_1^2)}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\Bigg\}\ (t_1<t<t_2),\end{aligned} \tag{16b}$$

$$\begin{aligned}\gamma s(t) &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma d_1 - \varepsilon\,\frac{E\kappa}{hc_1c_2}\left(v_0+2\dot f(t-t_2/2)\right)\\ &+\varepsilon\, d_1d_2 g(t-t_1)+d_2(\varepsilon+2d_1^2)\gamma s(t-t_1)\Bigg\},\quad (t_2<t<\frac{3}{2}t_1),\end{aligned} \tag{16c}$$

$$g(t)=\frac{1}{\Delta}\left\{\Theta_1\gamma - v_0E(1+d_1d_2)\frac{c_1+c_2}{c^2}\right\},\qquad (0<t<t_1), \tag{17a}$$

$$\begin{aligned}g(t) &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma + v_0E(1+d_1d_2)\frac{c_1+c_2}{c^2}\\ &+\frac{2\kappa}{hc_1(c_2^2-c_1^2)}\left[\varepsilon c^2 g(t-t_1)+(c^2-c_1^2)\gamma s(t-t_1)\right]\\ &-\frac{c_1(1+d_1d_2)}{c^2(c_2-c_1)}\left[(c^2-c_2^2)g(t-t_1)-c^2\gamma s(t-t_1)\right]\\ &+\frac{2E(1+d_1d_2)}{c^2(c_2-c_1)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right]\\ &-\varepsilon\,\frac{2E\kappa}{h(c_2^2-c_1^2)}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\Bigg\}\ (t_1<t<t_2),\end{aligned} \tag{17b}$$

$$\begin{aligned}g(t) &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma + E(1+d_1d_2)\frac{c_1+c_2}{c^2}\left(v_0+2\dot f(t-t_2/2)\right)\\ &+(1+d_1d_2)\left(d_1 g(t-t_1)-\gamma s(t-t_1)\right)\\ &+d_2\left(\varepsilon g(t-t_1)+d_1\gamma s(t-t_1)\right)\Bigg\},\quad (t_2<t<\frac{3}{2}t_1),\end{aligned} \tag{17c}$$

$$\dot f(t)=-v_0,\qquad (0<t<\frac{1}{2}t_1), \tag{18a}$$

$$\dot f(t)=-v_0+\frac{(c_1^2-a^2)g(t-t_1/2)-2c_1^2\gamma s(t-t_1/2)}{E(c_2-c_1)d_3},\quad (\frac{1}{2}t_1<t<\frac{1}{2}t_2), \tag{18b}$$

$$\dot{f}(t) = -v_0 - \frac{c_2 + c_1}{Ed_3}\left[g(t) - 2\gamma s(t - t_1/2)\right], \quad (\frac{1}{2}t_2 < t < t_1), \tag{18c}$$

where

$$\Delta = \varepsilon\, d_2 + d_1(1 + d_1 d_2),$$

$$d_1 = 1 + \frac{c_1 c_2}{c^2}, \quad d_2 = \frac{\kappa c^2}{h c_1 c_2 (c_1 + c_2)}, \quad d_3 = d_1 + \varepsilon.$$

Further it will be shown that the duration of contact of the rod with the wall cannot exceed the value of t_2. Because of this, there is no point in extending the functions $s(t)$ and $g(t)$ beyond the time interval $t = \frac{3}{2}t_1$.

The formulas (16)–(18) define the functions $\dot{f}(t)$, $s(t)$, and, $g(t)$ and together with the formulas (11), give us the solution of the problem under consideration. This solution is valid until the rod is in contact with the wall, *i.e.* as long as the contact stress $\sigma(0, t) < 0$.

To determine the contact time, let us investigate the time-dependence of the contact stress. For this purpose, write the contact stress in the Laplace domain

$$\bar{\sigma}(0, p) = E\bar{u},_x(0, p) - \gamma\bar{\theta}(0, p) = \bar{g} - \gamma\bar{s}. \tag{19}$$

Going in the formula (19) from the image to pre-image with due account for (16)–(18), we have

$$\sigma(0, t) = -\frac{E}{\Delta}\left\{\alpha\Theta_1 \frac{c_1 c_2}{c^2} + \frac{v_0(c_1 + c_2)}{c^2}(1 + d_2 d_3)\right\} < 0, \qquad (0 < t < t_1), \tag{20a}$$

$$\begin{aligned}
\sigma(0, t) = \frac{1}{\Delta}\Bigg\{&-\gamma\Theta_1\frac{c_1 c_2}{c^2} - \frac{v_0 E(c_1 + c_2)}{c^2}(1 + d_2 d_3) \\
&- \frac{c_1(1 + d_2 d_3)}{c^2(c_2 - c_1)}\left[(c^2 - c_2^2)g(t - t_1) - c^2\gamma s(t - t_1)\right] \\
&- \frac{2c_2\kappa}{c^2(c_2^2 - c_1^2)h}\left[\varepsilon c^2 g(t - t_1) + (c^2 - c_1^2)\gamma s(t - t_1)\right] \\
&- \frac{2}{c^2 d_3 (c_2 - c_1)^2}\left(c_1^2 - a^2 + \varepsilon\,\frac{c_1 c_2 \kappa}{(c_2 + c_1)h}\right) \\
&\times \left[(c_1^2 - a^2)g(t - t_1) - 2c_1^2\gamma s(t - t_1)\right]\Bigg\},
\end{aligned} \tag{20b}$$

$$(t_1 < t < \frac{1}{2}t_1 + \frac{1}{2}t_2),$$

$$\begin{aligned}\sigma(0,t) &= \frac{1}{\Delta}\left\{-\gamma\Theta_1\frac{c_1c_2}{c^2} + \frac{v_0E(c_1+c_2)}{c^2}(1+d_2d_3)\right.\\
&+\frac{2E(1+d_2d_3)}{c^2(c_2-c_1)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right]\\
&-\frac{c_1(1+d_2d_3)}{c^2(c_2-c_1)}\left[(c^2-c_2^2)g(t-t_1)-c^2\gamma s(t-t_1)\right]\\
&-\frac{2c_2\kappa}{c^2(c_2^2-c_1^2)h}\left[\varepsilon c^2 g(t-t_1)+(c^2-c_1^2)\gamma s(t-t_1)\right]\\
&\left.+\,\varepsilon\,\frac{2Ec_1c_2\kappa}{c^2(c_2^2-c_1^2)h}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\right\},\\
&\qquad (\frac{1}{2}t_1+\frac{1}{2}t_2<t<t_2),\end{aligned} \tag{20c}$$

$$\begin{aligned}\sigma(0,t) &= \frac{1}{\Delta}\left\{-\gamma\Theta_1\frac{c_1c_2}{c^2} - \frac{v_0E(c_1+c_2)}{c^2}(1+d_2d_3)\right.\\
&-\frac{2(c_1+c_2)^2}{c^2d_3}(1+d_2d_3)\left[g(t-t_1)-2\gamma s(t-t_1)\right]\\
&+(1+d_2d_3)\left[d_1g(t-t_1)-\gamma s(t-t_1)\right]\\
&\left.-\,\frac{2\kappa}{(c_2+c_1)h}\left[\varepsilon g(t-t_1)+d_1\gamma s(t-t_1)\right]\right\},\quad (t_2<t<\frac{3}{2}t_1).\end{aligned} \tag{20d}$$

If we neglect the coupling of the strain and temperature fields and put $\varepsilon = 0$, then the relationships (20) for the contact stress take the form

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma}{(1+d)(1+z_0)} < 0, \quad (0<t<t_1), \tag{21a}$$

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma[1+d(3-2z_0)]}{(1+d)^2(1-z_0^2)} < 0, \quad (t_1<t<\frac{1}{2}t_1+\frac{1}{2}t_2), \tag{21b}$$

$$\sigma(0,t) = -\frac{v_0E}{c} + \frac{\Theta_1\gamma[3+d(1+2z_0)]}{(1+d)^2(1-z_0^2)}, \quad (\frac{1}{2}t_1+\frac{1}{2}t_2<t<t_2), \tag{21c}$$

$$\sigma(0,t) = \frac{\Theta_1\gamma[3+d+2z_0(1+d)]}{(1+d)^2(1+z_0)} > 0, \quad (t_2<t<\frac{3}{2}t_1), \tag{21d}$$

where

$$d = \frac{\kappa}{ha}, \quad z_0 = \frac{c}{a}.$$

Reference to formulas (21) shows that the contact stress is governed by two terms: the first one depending on the rod's elastic properties is defined by the initial velocity of impact v_0, and the second one depending on the thermal properties is dictated by the temperature of the wall's heating Θ_1. Analysis of the relationships (21) shows that, prior to the instant $t = t_1$, the contact stress is less than zero; at the moment $t = t_1$ (the moment of arrival of the reflected thermal T_T-wave at the place of contact) it changes abruptly remaining a negative constant value up to the instant $t = \frac{1}{2}t_1 + \frac{1}{2}t_2$ (the moment of concurrent arrival of two reflected waves, T_E- and E_T-waves, at the place of contact). In other words, prior to the moment of arrival of the wave of off-loading at the place of contact, *i.e.* the elastic E_T-wave reflected from the incident thermal T-wave, both terms have the same sign, and the recoil of the rod does not occur. The thermal wave T_T arriving at the place of contact at the moment $t = t_1$ only loads additionally the place of contact. At the moment of concurrent arrival of two reflected waves, T_E- and E_T-waves, at the place of contact, the second term in (21) changes its sign from '$-$' to '$+$'. Since the sign of the first term in (21) remains unchanged, the rebound of the rod from the wall can occur only at a certain magnitude of the value v_0/Θ_1, *i.e.* depending on what kind of the processes predominates: thermal or elastic. Thus, when $v_0/c\alpha\Theta_1 \leq \nu$ the rebound will take place at the instant $t = \frac{1}{2}t_1 + \frac{1}{2}t_2$, where

$$\nu = \frac{3 + d(1 + 2z_0)}{(1 + d)^2(1 - z_0^2)}.$$

If $v_0/c\alpha\Theta_1 > \nu$, the recoil will occur at the instant $t = t_2$, when the reflected elastic E_E-wave of off-loading arrives at the place of contact (the off-loading E_E-wave is generated from the action of the incident elastic E-wave of loading onto the rod's free end), and the first term in (21) vanishes, but the second one remains positive.

Note that the coupling of strain and temperature fields is small for many materials (this is valid for the majority of metals), and hence the dimensionless parameter ε characterizing this coupling is a small value (for metals the value of ε has the order of 10^{-2}, see [16]). Therefore the conclusions made above for the behaviour of the contact stress without reference to the coupling of strain and temperature fields, *i.e.* when $\varepsilon = 0$, remain valid for the materials possessing small magnitudes of ε. But for other materials Equations (20) require numerical investigation.

3.2. THE CASE $t_1 > t_2$

Now assume that $t_1 > t_2$ (see the second example in Figure 1). In this case, the formulas (16)–(18) take the form

$$\gamma s(t) = \frac{1}{\Delta}\left\{\Theta_1 \gamma d_1 + \varepsilon\, v_0 E\, \frac{\kappa}{hc_1c_2}\right\}, \quad (0 < t < t_2), \tag{22a}$$

$$\begin{aligned}
\gamma s(t) \ &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma d_1 - \varepsilon\ v_0 E\ \frac{\kappa}{hc_1c_2} \\
&-\frac{2d_1\kappa}{hc_2(c_2^2-c_1^2)}\left[\varepsilon c^2 g(t-t_2)+(c^2-c_2^2)\gamma s(t-t_2)\right] \\
&-\varepsilon\frac{\kappa}{hc_1(c_2^2-c_1^2)}\left[(c^2-c_1^2)g(t-t_2)-c^2\gamma s(t-t_2)\right] \\
&-\varepsilon\ \frac{2E\kappa}{hc_1c_2(c_2^2-c_1^2)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right] \\
&-\ \varepsilon\ \frac{2Ed_1\kappa}{h(c_2^2-c_1^2)}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\Bigg\},\ (t_2<t<\frac{1}{2}t_2+\frac{1}{2}t_1),
\end{aligned} \tag{22b}$$

$$g(t)=\frac{1}{\Delta}\left\{\Theta_1\gamma - v_0E(1+d_1d_2)\frac{c_1+c_2}{c^2}\right\}, \qquad (0<t<t_2), \tag{23a}$$

$$\begin{aligned}
g(t) \ &= \frac{1}{\Delta}\Bigg\{\Theta_1\gamma + v_0E(1+d_1d_2)\frac{c_1+c_2}{c^2} \\
&-\frac{2\kappa}{hc_2(c_2^2-c_1^2)}\left[\varepsilon c^2 g(t-t_2)+(c^2-c_2^2)\gamma s(t-t_2)\right] \\
&+\frac{c_2(1+d_1d_2)}{c^2(c_2-c_1)}\left[(c^2-c_1^2)g(t-t_2)-c^2\gamma s(t-t_2)\right] \\
&+\frac{2E(1+d_1d_2)}{c^2(c_2-c_1)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right] \\
&-\ \varepsilon\ \frac{2E\kappa}{h(c_2^2-c_1^2)}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\Bigg\}, \quad (t_2<t<\frac{1}{2}t_2+\frac{1}{2}t_1),
\end{aligned} \tag{23b}$$

$$\dot f(t) = -v_0, \qquad (0<t<\frac{1}{2}t_2), \tag{24a}$$

$$\dot f(t) = -v_0 - \frac{(c_2^2-a^2)g(t-t_2/2)-2c_2^2\gamma s(t-t_2/2)}{E(c_2-c_1)d_3}, \quad (\frac{1}{2}t_2<t<\frac{1}{2}t_1). \tag{24b}$$

Instead of the formulas (20) we have

$$\sigma(0,t) = -\frac{E}{\Delta}\left\{\alpha\Theta_1\frac{c_1c_2}{c^2}+\frac{v_0(c_1+c_2)}{c^2}(1+d_2d_3)\right\} < 0, \quad (0<t<t_2), \tag{25a}$$

$$\begin{aligned}\sigma(0,t) &= \frac{1}{\Delta}\left\{-\gamma\Theta_1\frac{c_1c_2}{c^2} + \frac{v_0E(c_1+c_2)}{c^2}(1+d_2d_3)\right.\\ &+\frac{2E(1+d_2d_3)}{c^2(c_2-c_1)}\left[(c_2^2-a^2)\dot f(t-t_2/2)-(c_1^2-a^2)\dot f(t-t_1/2)\right]\\ &+\frac{c_2(1+d_2d_3)}{c^2(c_2-c_1)}\left[(c^2-c_1^2)g(t-t_2)-c^2\gamma s(t-t_2)\right]\\ &+\frac{2c_1\kappa}{c^2(c_2^2-c_1^2)h}\left[\varepsilon c^2 g(t-t_2)+(c^2-c_2^2)\gamma s(t-t_2)\right]\\ &\left.+\,\varepsilon\,\frac{2Ec_1c_2\kappa}{c^2(c_2^2-c_1^2)h}\left[\dot f(t-t_2/2)-\dot f(t-t_1/2)\right]\right\},\ (t_2<t<\tfrac{1}{2}t_1+\tfrac{1}{2}t_2).\end{aligned} \tag{25b}$$

If we neglect the coupling of the strain and temperature fields and put $\varepsilon = 0$, then the relationships (25) for the contact stress take the form

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma}{(1+d)(1+z_0)} < 0, \quad (0<t<t_2), \tag{26a}$$

$$\sigma(0,t) = \frac{\Theta_1\gamma(1+2z_0^2)}{(1+d)(z_0^2-1)} > 0, \quad (t_2<t<\tfrac{1}{2}t_1+\tfrac{1}{2}t_2). \tag{26b}$$

Reference to the formulas (26) shows that at $t_1 > t_2$ the duration of contact is equal to t_2, as well as for the materials possessing small magnitudes of ε, but for other materials Equations (25) require numerical investigation.

Note that, if c_2 is much larger than c_1, the rebound of the rod will occur at the instant $t = t_2$ when the quasielastic qE_{qE}-wave reaches the place of contact. If c_1 is much larger than c_2, the rebound of the rod will occur at the moment $t = t_1/2 + t_2/2$ or $t = t_2$ when the quasielastic qE_{qT}-wave or qE_{qE}-wave reaches the place of contact, respectively, since only the quasielastic waves have the character of the off-loading wave. If $c_1 \to \infty$, the elastic wave will propagate along the disturbed rod, and its recoil will take place at the moment $t = t_2$.

3.3. Particular cases

3.3.1. *Perfect thermal contact*

It should be noted that, if the perfect thermal contact is established during impact of the rod against the rigid wall, *i.e.* $h \to \infty$ and hence $d_2 = 0$ and $\Delta = d_1$, then the time-dependence of the contact stress has the form:
for $t_1 < t_2$ when $\varepsilon \neq 0$

$$\sigma(0,t) = -\frac{1}{d_1}\left(v_0E\frac{c_1+c_2}{c^2} + \Theta_1\gamma\frac{c_1c_2}{c^2}\right) < 0, \quad (0<t<t_1), \tag{27a}$$

$$\begin{aligned}\sigma(0,t) &= \frac{1}{d_1^2}\left\{\left(v_0E\frac{c_1+c_2}{c^2}+\Theta_1\gamma\frac{c_1c_2}{c^2}\right)\left[\frac{c_1}{c_2-c_1}-d_1+\frac{2c_1^2(c_1^2-a^2)}{c^2d_3(c_2-c_1)^2}\right]\right.\\ &+\left(\Theta_1\gamma - v_0E\frac{c_1+c_2}{c^2}\right)\left[\frac{c_1c_2^2}{c^2(c_2-c_1)}+\frac{2a^2(c_1^2-a^2)}{c^2d_3(c_2-c_1)^2}\right]\\ &\left.+\frac{2c_1d_1(c_1^2-a^2)}{c^2d_3(c_2-c_1)^2}\Theta_1\gamma\right\}, \quad (t_1<t<\tfrac{1}{2}t_1+\tfrac{1}{2}t_2).\end{aligned} \tag{27b}$$

$$\sigma(0,t) = \frac{1}{d_1^2}\left\{\left(v_0 E\frac{c_1+c_2}{c^2} + \Theta_1\gamma\frac{c_1c_2}{c^2}\right)\left(\frac{c_1}{c_2-c_1} - d_1\right)\right.$$
$$+\left(\Theta_1\gamma - v_0E\frac{c_1+c_2}{c^2}\right)\left[\frac{c_1c_2^2}{c^2(c_2-c_1)} + \frac{2(2c_2^2-c_1^2-a^2)(c_1^2-a^2)}{c^2d_3(c_2-c_1)^2}\right] \tag{27c}$$
$$\left.- \frac{4d_1[c_1^2(c_2^2-c_1^2)+c_2^2(c_1^2-a^2)]}{c^2d_3(c_2-c_1)^2}\Theta_1\gamma\right\}, \quad (\frac{1}{2}t_1 + \frac{1}{2}t_2 < t < t_2),$$

$$\sigma(0,t) = \frac{1}{d_1}\left\{\Theta_1\gamma\left[\frac{2(c_1+c_2)^2(2d_1-1)}{c^2d_1d_3} - \frac{c_1c_2}{c^2}\right] - 2\varepsilon v_0E\frac{c_1c_2(c_1+c_2)}{c^4d_1d_3}\right\} \tag{27d}$$
$$(t_2 < t < \frac{3}{2}t_1),$$

and for the case $\varepsilon = 0$

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma}{1+z_0} < 0, \qquad (0 < t < t_1), \tag{28a}$$

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma}{1-z_0^2} < 0, \qquad (t_1 < t < \frac{1}{2}t_1 + \frac{1}{2}t_2), \tag{28b}$$

$$\sigma(0,t) = -\frac{v_0E}{c} + \frac{3\Theta_1\gamma}{1-z_0^2}, \qquad (\frac{1}{2}t_1 + \frac{1}{2}t_2 < t < t_2) \tag{28c}$$

$$\sigma(0,t) = \frac{\Theta_1\gamma(3+2z_0)}{1+z_0} > 0, \qquad (t \geq t_2) \tag{28d}$$

for $t_1 > t_2$ when $\varepsilon \neq 0$

$$\sigma(0,t) = -\frac{1}{d_1}\left(v_0E\frac{c_1+c_2}{c^2} + \Theta_1\gamma\frac{c_1c_2}{c^2}\right) < 0, \qquad (0 < t < t_2), \tag{29a}$$

$$\sigma(0,t) = \frac{1}{d_1^2}\left\{v_0E\frac{c_1+c_2}{c^2}\left[\frac{c_2(c_1^2-c^2)}{c^2(c_2-c_1)} - d_1 + \frac{2(c_2^2-a^2)^2}{c^2d_3(c_2-c_1)^2}\right]\right.$$
$$\left.+ \Theta_1\gamma\left[\frac{2(c_2^2-a^2)(2d_1c_2^2-c_2^2+a^2)}{c^2d_3(c_2-c_1)^2} - \frac{c_1c_2}{c^2}\left(d_1 + \frac{c_1+c_2}{c_2-c_1}\right)\right]\right\}, \tag{29b}$$
$$(t_2 < t < \frac{1}{2}t_2 + \frac{1}{2}t_1)$$

and for the case $\varepsilon = 0$

$$\sigma(0,t) = -\frac{v_0E}{c} - \frac{\Theta_1\gamma}{1+z_0} < 0, \qquad (0 < t < t_2), \tag{30a}$$

$$\sigma(0,t) = \frac{\Theta_1\gamma(1+2z_0^2)}{z_0^2-1} > 0, \qquad (t \geq t_2). \tag{30b}$$

3.3.2. *The case of thermally isolated rod's cross-section being in contact with a wall*
If the rod's cross-section, which is in contact with the wall, is thermally insulated, *i.e.* $h \to 0$ and hence $d_2 \to \infty$ and $\Delta/d_2 = \varepsilon + d_1^2$, then the time-dependence of the contact stress has the form:
for $t_1 < t_2$ when $\varepsilon \neq 0$

$$\sigma(0,t) = -v_0 E \frac{c_1+c_2}{c^2} \frac{d_3}{\varepsilon + d_1^2} < 0, \qquad (0 < t < t_1), \tag{31a}$$

$$\begin{aligned} \sigma(0,t) &= \frac{v_0 E}{(\varepsilon+d_1^2)^2} \frac{c_1+c_2}{c^2} \left\{ d_1 d_3 \left[\frac{c_1(c^2-c_2^2)}{c^2(c_2-c_1)} - d_1 \right] \right. \\ &\quad \left. + \varepsilon \left[\frac{d_3(2c_1-c_2)}{c_2-c_1} + \frac{2a^2(c_2+c_1)}{c^2} + \frac{2a^2[d_1(c_1^2-a^2)+2\varepsilon c_1^2]}{c^2 d_3 (c_2-c_1)^2} \right] \right\}, \\ &\qquad (t_1 < t < \frac{1}{2}t_1 + \frac{1}{2}t_2), \end{aligned} \tag{31b}$$

$$\begin{aligned} \sigma(0,t) &= \frac{v_0 E}{(\varepsilon+d_1^2)^2} \frac{c_1+c_2}{c^2} \left\{ -d_3(\varepsilon+d_1^2) + \frac{c_1 d_3 (d_3 c^2 - d_1 c_2^2)}{c^2(c_2-c_1)} \right. \\ &\quad - \frac{2(d_3c_1^2 - d_1a^2)(d_3+\varepsilon)(c_1+c_2)}{c^2 d_3 (c_2-c_1)} + \varepsilon \frac{2c_1c_2^2[c^2(d_1+1)-c_1^2]}{c^4(c_2-c_1)} \\ &\quad \left. - \frac{2(d_3c_2^2 - d_1a^2)[d_1(c_1^2-a^2)+2\varepsilon c_1^2]}{c^2 d_3 (c_2-c_1)^2} \right\}, \qquad (\frac{1}{2}t_1 + \frac{1}{2}t_2 < t < t_2), \end{aligned} \tag{31c}$$

$$\sigma(0,t) = \frac{2v_0 E}{(\varepsilon+d_1^2)} \frac{c_1+c_2}{c^2} \left[-d_3 + \frac{(c_1+c_2)^2(d_3+\varepsilon)}{c^2(d_1^2+\varepsilon)} \right], \qquad (t_2 < t < \frac{3}{2}t_1) \tag{31d}$$

and for the case $\varepsilon = 0$

$$\sigma(0,t) = -\frac{v_0 E}{c} < 0, \qquad (0 < t < t_2), \tag{32a}$$

$$\sigma(0,t) = 0, \qquad (t \geq t_2) \tag{32b}$$

for $t_1 > t_2$ at $\varepsilon \neq 0$

$$\sigma(0,t) = -v_0 E \frac{c_1+c_2}{c^2} \frac{d_3}{\varepsilon + d_1^2} < 0, \qquad (0 < t < t_2), \tag{33a}$$

$$\begin{aligned} \sigma(0,t) &= \frac{v_0 E}{(\varepsilon+d_1^2)^2} \frac{c_1+c_2}{c^2} \left\{ -d_3(\varepsilon+d_1^2) - \frac{c_2 d_3 (d_3 c^2 - d_1 c_1^2)}{c^2(c_2-c_1)} \right. \\ &\quad + \frac{2[(d_3+\varepsilon)c_2^2 - d_1a^2][d_3c_2^2 + a^2(\varepsilon - d_1)]}{c^2 d_3 (c_2-c_1)^2} \\ &\quad \left. - \varepsilon \frac{2a^2(c_1+c_2)}{c^2(c_2-c_1)} \right\}, \qquad (t_2 < t < \frac{1}{2}t_2 + \frac{1}{2}t_1) \end{aligned} \tag{33b}$$

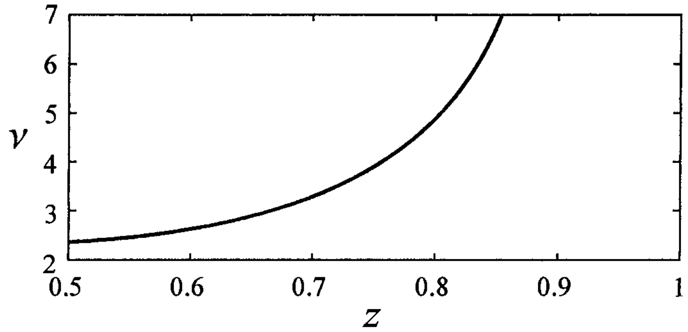

Figure 2. The z_0-dependence of the value ν at $d = 0{\cdot}5$.

and for the case $\varepsilon = 0$ Equations (33) go over into Equations (32).

3.3.3. *The impact of an elastic rod*

In order to obtain the known solution of the problem of impact of an elastic rod against a rigid barrier [17, Chapter 3], it is necessary to ignore thermal processes occurring in the rod, *i.e.* to put $\alpha = 0$ in the solution found above. Then only the first term remains in the relationship for $u(x, t)$ and the time-dependence of the contact stress has the form

$$\sigma(0, t) = -\frac{v_0 E}{c}, \qquad (0 < t < t_2), \tag{34a}$$

$$\sigma(0, t) = 0, \qquad (t \geq t_2). \tag{34b}$$

4. Numerical results

Numerical investigations are centred on the analysis of the contact stress. For this purpose we introduce the following dimensionless values:

$$\sigma^*(0, t^*) = \frac{\sigma(0, t)}{E}, \quad v_0^* = \frac{v_0}{c_2}, \quad \Theta_1^* = \alpha\Theta_1, \quad z = \frac{t_1}{t_2} = \frac{c_2}{c_1}, \quad t^* = \frac{t}{t_2}.$$

Figure 2 shows the z_0-dependence of the value ν at $d = 0{\cdot}5$. From Figure 2 it is seen that the function $\nu(z_0)$ tends monotonically to infinity as $z_0 \to 1$.

The t^*-dependence of the contact stress σ^*/Θ_1^* is presented in Figures 3a and 3b for $z = 0{\cdot}6$ and $z = 1{\cdot}4$, respectively, at $v_0^*/\Theta_1^* = 1$ and $d = 0{\cdot}5$. Reference to Figures 3 shows that taking account of coupling the temperature and strain fields ($\varepsilon \neq 0$) results in an increase of the contact-stress magnitude as compared with the case of the uncoupled problem of thermoelasticity ($\varepsilon = 0$). It should be noted also that the contact stress, as in the elastic case, remains unchanged during the time instants between arrivals of the reflected waves at the place of contact, *i.e.* the contact stress does not relax as time goes on.

Figure 4 shows the duration of contact t^*_{cont} as a function of the value v_0^*/Θ_1^* at $z = 0{\cdot}6, 0{\cdot}7$ and $0{\cdot}8$. From Figure 4 it is seen that, during the transition through $v_0^*/\Theta_1^* = \nu$, the duration of contact changes abruptly and that the inclusion of heat exchange between the rod and the wall leads to the decrease in the duration of contact under certain magnitudes of v_0^*/Θ_1^*. The value v_0^*/Θ_1^* by itself increases monotonically as z increases.

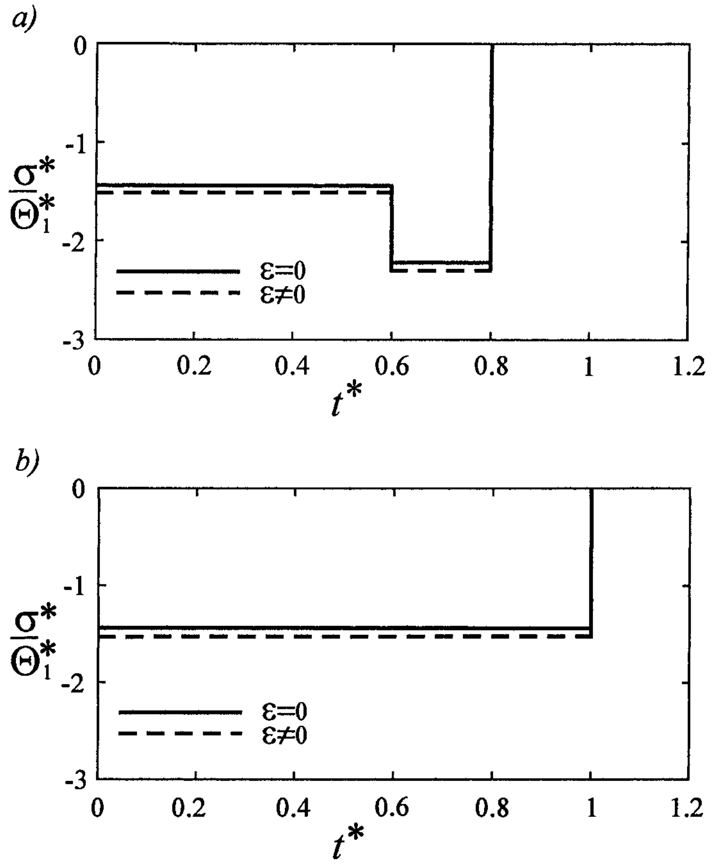

Figure 3. The t^*-dependence of the contact stress σ^*/Θ_1^* for (a) $z = 0{\cdot}6$ and (b) $z = 1{\cdot}4$ at $v_0^*/\Theta_1^* = 1$, $\varepsilon = 0{\cdot}03$, and $d = 0{\cdot}5$.

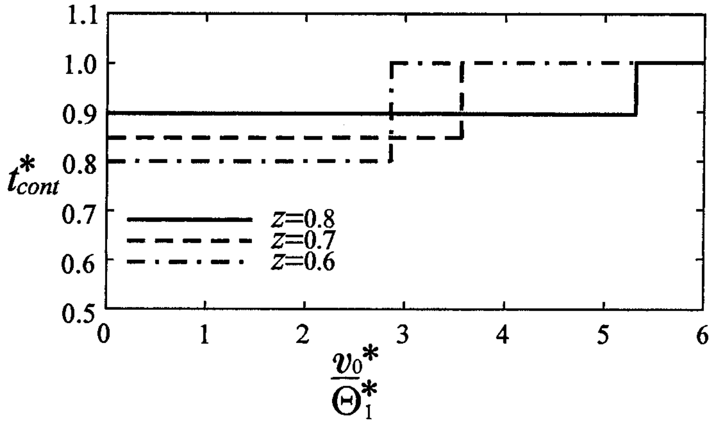

Figure 4. The v_0^*/Θ_1^*-dependence of the duration of contact t_{cont}^*.

5. Conclusions

From the above discussion the following conclusions can be drawn:

(1) The problem of the impact of a thermoelastic rod against a rigid heated barrier with due account for heat exchange between the rod and the wall has been analytically solved exactly for the first time.

(2) The influence of the coupling of the strain and temperature fields on the time-dependence of the contact stress has been analyzed: the fields' coupling results in an increase in the contact stress.

(3) Since the contact stress is the sum of two terms, one of which is determined by the mechanical excitation and depends upon the initial velocity of impact, and the other term is governed by the thermal excitation and depends upon heating temperature of the wall, the duration of contact between the rod and the wall may vary depending on the magnitudes of these two terms.

Appendix A. Fourier series for the generalized functions

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\cos(2n-1)\frac{\pi t}{t_i} = \frac{t_i}{\pi}\delta(t) + \frac{2t_i}{\pi}\sum_{n=1}^{\infty}(-1)^n\delta(t-nt_i), \tag{A1}$$

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sin(2n-1)\frac{\pi t}{t_i} = \frac{2t_i}{\pi}\sum_{n=1}^{\infty}(-1)^n\delta\left[t-\left(n-\frac{1}{2}\right)t_i\right], \tag{A2}$$

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}(2n-1)\sin(2n-1)\frac{\pi t}{t_i} = \left(\frac{t_i}{\pi}\right)^2\delta'(t) + 2\left(\frac{t_i}{\pi}\right)^2\sum_{n=1}^{\infty}(-1)^n\delta'(t-nt_i), \tag{A3}$$

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)\cos(2n-1)\frac{\pi t}{t_i} = 2\left(\frac{t_i}{\pi}\right)^2\sum_{n=1}^{\infty}(-1)^n\delta'\left[t-\left(n-\frac{1}{2}\right)t_i\right], \tag{A4}$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos[(2n-1)\pi t t_i^{-1}]}{2n-1} = \begin{cases} 1, & 0 < t < \frac{1}{2}t_i \\ -1, & \frac{1}{2}t_i < t < \frac{3}{2}t_i \end{cases}, \tag{A5}$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi t t_i^{-1}]}{2n-1} = \begin{cases} 1, & 0 < t < t_i \\ -1, & t_i < t < 2t_i \end{cases}, \tag{A6}$$

where $\delta(x)$ is the Dirac δ-function, and $t_i = 2l/c_i$ $(i = 1, 2)$.

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